Density Stratified, Viscous Flow past a Flat Plate

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Analytical approximations of the stratified, laminar flow past a flat plate are considered. Perturbation solutions are obtained for the flow on the plate and in the far wake with a boundary-layer type of approximation and for the whole flowfield with a low Reynolds' number approximation. The boundary-layer approximate solutions show that the wall shear on the plate is increased and that the rate of decay of the velocity defect in the wake is markedly increased. In the low Reynolds' number regime, the effects of the stratification are much larger ahead of and behind the plate than on the plate itself.

Nomenclature

a	= distance behind plate trailing edge
B	$\equiv gk/CU_e^2$
C	$= (2)^{1/2} - 1$
C_f	= skin-friction coefficient
C_{f0}	= skin-friction coefficient at the same station with no stratification
$f(x,y)$	= nonhomogeneous term in Eq. (18)
$\bar{F}_0'$	= flow variable from Ref. 3
g	= gravity
h	= step size in numerical calculations
H	= distance to lateral boundary
k	= measure of freestream stratification
L	= length of the plate
$M+1$	= number of grid points in the streamwise direction
$N+1$	= number of grid points in the normal direction
Re	$\equiv U_e/L/\nu$
Ri	$\equiv gkL^2/U_e^2$
s	= salinity
u	= streamwise velocity
U_e	= freestream velocity
x	= streamwise coordinate
X	$\equiv (x/L)\,Re^{-1}$
y	= normal coordinate
Y	$\equiv y/L$
ψ	= steam function
ν	= kinematic viscosity
ϵ	= parameter in salinity-density relation
ρ	= density
$\bar{\eta}$	= coordinate in Ref. 3
ξ	$\equiv \nu x/CU_e$
β	$\equiv gk/U_e\nu$
Ω	= dummy variable
δ	= error limit
α	$\equiv (a/L)\,Re^{-1}$

Superscripts

(0), (1), (2) = order of the variable in the perturbation solution

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Introduction

THE flow of a density stratified fluid past an obstacle is of interest in many practical situations. Among these are problems in the atmosphere where the density variations are caused primarily by temperature and in the ocean where the stratification results from both temperature and salinity variations.

Stratified flows exhibit important "inviscid" effects such as internal waves which tend to make the region of strong influence of an obstacle much larger than for a corresponding case with a homogeneous fluid. Yih[1] has given an extensive review of this aspect of these problems, but less work has been presented dealing directly with viscous, stratified flow problems. References 2 and 3 are representative of the studies to date. In the present work, we shall attempt to advance the state of analyses of viscous, stratified flows by considering the planar, laminar flow past a flat plate in detail. Some of the more untenable assumptions employed in previous works will be relieved, and comparisons between the same flow problem with and without stratification will be used to illuminate the effects of the stratification.

We begin by considering the basis of the existing treatments. Using a standard, planar coordinate system as illustrated in Fig. 1 and taking a linearly stratified $(1/\rho_0)(\partial\rho/\partial y) \equiv -k$), incompressible flow with a uniform velocity, U_e, approaching a flat plate, the equations of motion can be written

$$[(\partial\psi/\partial y)(\partial/\partial x) - (\partial\psi/\partial x)(\partial/\partial y) - \nu\nabla^2]\nabla^2\psi = g\epsilon(\partial s/\partial x) \tag{1}$$

$$(\partial\psi/\partial y)(\partial s/\partial x) - (\partial\psi/\partial x)(\partial s/\partial y) = D\nabla^2 s \tag{2}$$

Here, ϵ is defined by the equation

$$\rho = \rho_0[1 + \epsilon(s - s_0)] \tag{3}$$

where s_0 and ρ_0 are mean quantities. The stratification under consideration here is due solely to salinity variations, and Martin and Long[2] have shown that, due to the smallness of the ratio $(D/\nu) \sim 10^{-3}$, it is reasonable to neglect diffusion. This yields the simple result

$$s(x,y) = (-k/\epsilon U_e)\psi(x,y) \tag{4}$$

by evaluating the uniform region far upstream. Thus, Eq.

(2) need no longer be treated and Eq. (1) becomes

$$[(\partial\psi/\partial y)(\partial/\partial x) - (\partial\psi/\partial x)(\partial/\partial y) - \nu\nabla^2]\nabla^2\psi = -(gk/U_e)\partial\psi/\partial x \quad (1a)$$

We must pause here to note that this equation is not the same as that derived by either Martin and Long[2] or Pao[3] even though they were all derived on apparently the same basis. In particular, the sign on the term reflecting the influence of the stratification is opposite here to that derived by the previous workers. This discrepancy is easily explained, and, indeed, the matter is one of considerable importance as we shall see below. Martin and Long[2] used a curious, nonstandard coordinate system (see their Fig. 5) wherein the axial velocity is taken as positive in the negative axial direction, i.e., u is positive in the $-x$ direction. Furthermore, the definition of the stream function $\psi(x,y)$ was taken as opposite in sign to that normally employed. Pao[3] arrived at the same result as Martin and Long by taking the freestream velocity as equal to $-U_e$. While it is possible to work with a system of equations developed in these unusual ways, it is at best confusing and did lead to conceptual errors on the part of the previous authors that will be illuminated momentarily. It is important to note here, however, that Eq. (1a) was developed using standard notation and sign conventions and is exact save for the nondiffusive assumption which is not brought into question by the discussion above.

It is clear that the solution of Eq. (1a) represents an extremely formidable problem, and some simplifications must be sought in order to render the situation tractable. This equation is, of course, the vorticity equation developed from the incompressible Navier-Stokes equations extended to include the influence of stratification. It is natural then to look for simplifications of the low Reynolds' number Stokes' or Oseen, type or high Reynolds' number, Prandtl boundary layer, type that have proven useful for unstratified flows. In a strong drive towards simplification, Martin and Long[2] have employed both types of approximation at the same time! First, the $\nabla^2(\nabla^2\psi)$ term was approximated as simply $\partial^4\psi/\partial y^4$ with a boundary-layer assumption, and then the whole convective derivative term was neglected by invoking a Stokes' flow assumption that inertia effects are negligible. Nowhere else in the literature have these two types of approximation been simultaneously invoked. Furthermore, the over-simplified equation thus derived does not reduce to a rational approximation of the homogeneous flow past a flat plate when $k \to o$ for any Reynolds' number regime. Pao,[3] however, worked with same equation.

The confusing coordinate systems used by the previous authors took their toll not in the form of the equation that was derived and used but in the matter of solving that equation. A "similar solution" approach was adopted where the nondimensional velocity field $\bar{u}(x,y)$ was solved in the form

$$\bar{u}(x,y) = \bar{F}_0'(\bar{\eta}) \quad (5)$$

with

$$\bar{\eta} \equiv \bar{y}/\bar{x}^{1/4} \quad (5a)$$

Since the "trailing" edge of the plate is located at $x = 0$ in the coordinate system used, the boundary conditions expressed in terms of the similarity variable, $\bar{\eta}$, require that there be a uniform velocity profile (i.e., a zero boundary-layer thickness) at the trailing edge. This results since $y \to \infty$ is the same as $x \to 0$ as far as the similarity variable, $\bar{\eta}$, is concerned. Since one wants the velocity to go to the uniform freestream value as $y \to \infty$, unavoidably one also gets the same behavior at $x \to 0$. In the standard coordinate system, this presents no difficulty since $x = 0$ is at the leading edge, and a uniform velocity profile at that point is in agreement with physical reality. However, it is clear that the peculiar result that the boundary layer decreases in size along the plate as presented in Refs. 2 and 3 is due primarily to the manner in which the mathematical problem was posed and solved therein.

In this paper, we shall develop approximations to Eq. (1a) of both the boundary layer and Stokes' flow type. The flow in the vicinity of a flat plate will be treated with each approximation. Finally, the far wake behind a flat plate will be analyzed using the boundary-layer form of the equation.

The Flowfield on the Plate

Boundary-Layer Regime

For high Reynolds' numbers, we introduce first a boundary-layer approximation, i.e., $\partial^n/\partial y^n \gg \partial^n/\partial x^n$ and Eq. (1a) becomes

$$(\partial\psi/\partial y)(\partial^3\psi/\partial x\partial y^2) - (\partial\psi/\partial x)(\partial^3\psi/\partial y^3) - \nu(\partial^4\psi/\partial y^4) = -(gk/U_e)(\partial\psi/\partial x) \quad (1c)$$

This equation remains mathematically difficult since the convective derivative terms on the left-hand side are nonlinear. For similar flow problems without stratification (note that the nonlinear terms are not explicitly affected by the stratification), it has been found that good approximations to exact solutions are obtained by the following linearization[4]

$$(\partial\psi/\partial y)(\partial/\partial x) - (\partial\psi/\partial x)(\partial/\partial y) \simeq CU_e(\partial/\partial x) \quad (6)$$

The factor C is determined by forcing the solution to satisfy some external condition. For a uniform, homogeneous flow past a flat plate, the use of the momentum integral equation as a condition, for example, results in $C = [(2)^{1/2} - 1]$ which gives an excellent approximation to the exact Blasius result.[4] We shall employ this approximation with the same value of C here to evaluate the effects of stratification on the same basic flow problem. This approach is taken as a first approximation; a further refinement would involve the redetermination of C for each stratified case. The equation to be solved is now reduced to

$$CU_e(\partial^2\psi/\partial y^2\partial x) + (gk/U_e)(\partial\psi/\partial x) = \nu(\partial^4\psi/\partial y^4) \quad (1d)$$

It is convenient to introduce a new axial coordinate, $\xi(x)$, such that $\xi(0) = 0$ and

$$\partial/\partial\xi = (CU_e/\nu)(\partial/\partial x) \quad (7)$$

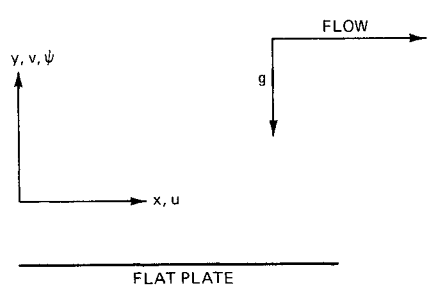

Fig. 1 Coordinate system and conventions.

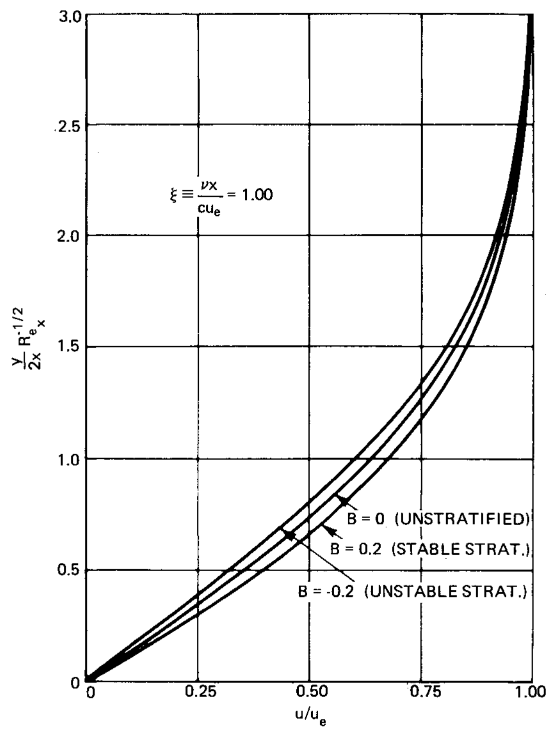

Fig. 2 Effects of stratification on velocity profiles as predicted by the boundary-layer analysis.

that is

$$\xi = (\nu/CU_e)x \tag{8}$$

Now, Eq. (1d) becomes

$$(\partial^3\psi/\partial y^2\partial\xi) + B(\partial\psi/\partial\xi) = (\partial^4\psi/\partial y^4) \tag{9}$$

where, $B \equiv gk/CU_e^2$.

Since our purpose here is primarily to illustrate the new effects of density stratification on viscous flows, it is useful as well as mathematically helpful to consider solutions to the stratified flow problem as perturbations upon the corresponding unstratified flow. Thus, we take

$$\psi(\xi,y) = \psi^{(0)}(\xi,y) + B\psi^{(1)}(\xi,y) + \frac{B^2}{2!}\,\psi^{(2)}(\xi,y) + \ldots \tag{10}$$

Substituting into Eq. (9) and collecting terms of equal order in the perturbation parameter, B, there results

$$0[B^\circ] \qquad [\partial^3\psi^{(0)}/\partial y^2\partial\xi] = \partial^4\psi^{(0)}/\partial y^4 \tag{9a}$$

$$0[B^1] \qquad [\partial^3\psi^{(1)}/\partial y^2\partial\xi] + \partial\psi^{(0)}/\partial\xi = \partial^4\psi^{(1)}/\partial y^4 \tag{9b}$$

The boundary conditions pertinent to the flat plate problem are

$$x = \xi = 0 \qquad \psi = U_e y \quad \psi^{(0)} = U_e y \quad \psi^{(1)} = 0$$

$$y = 0 \qquad \psi = 0 \quad \psi^{(0)} = 0 \quad \psi^{(1)} = 0$$

$$\frac{\partial\psi}{\partial y} = 0 \quad \partial\psi^{(0)}/\partial y = 0 \quad \partial\psi^{(1)}/\partial y = 0$$

$$\lim \text{ as } y \to \infty \qquad \psi = U_e y \quad \psi^{(0)} = U_e y \quad \psi^{(1)} = 0$$

The Eqs. (9a), (9b), etc. can all be treated using LaPlace transforms. Equation (9a) describes a simple, unstratified flow, and with the boundary conditions here, the solution is[4]

$$\psi^{(0)}(\xi,y) = U_e y - 2U_e(\xi/\pi)^{1/2} + 2U_e(\xi)^{1/2}\,\text{ierfc}\,[y/(4\xi)^{1/2}] \tag{11}$$

From this, the nonhomogeneous term in Eq. (9b) can be found as

$$\partial\psi^{(0)}/\partial\xi = \frac{1}{(\xi)^{1/2}} \left[\left\{\frac{y}{(4\xi)^{1/2}}\right\}\text{erfc}\left\{\frac{y}{(4\xi)^{1/2}}\right\} - \frac{1}{(\pi)^{1/2}} + \text{ierfc}\left\{\frac{y}{(4\xi)^{1/2}}\right\}\right] \tag{11a}$$

The solution to Eq. (9b) with this term can then be determined, and it is given by

$$\psi^{(1)}(\xi,y) = -2U_e\xi\left\{\frac{2}{3}\left[\frac{y^2}{4\xi}\right]\text{erfc}\left[\frac{y}{(4\xi)^{1/2}}\right] - \frac{2}{3(\pi)^{1/2}}\left[\frac{y}{(4\xi)^{1/2}}\right]e^{-y^2/4\xi} + \frac{2}{3}\,\text{i}^2\text{erfc}\left[\frac{y}{(4\xi)^{1/2}}\right] - \frac{1}{6}\,\text{erfc}\left[\frac{y}{(4\xi)^{1/2}}\right]\right\} \tag{12}$$

Successive approximations can be found in the same way, but it will suffice here to show some graphical results based only on the first order perturbation in Fig. 2. Of course, valid results are limited to the condition, $B\psi^{(1)}(\xi,y) \gg B^2\psi^{(2)}(\xi,y)$, which generally requires $B \equiv gk/CU_e^2 \ll 1$.

The results in Fig. 2 show that the effects of a stable density stratification, k and therefore $B > 0$, are such as to produce a fuller velocity profile. This prediction is confirmed by the experimental results of Ref. 3. If one wishes to consider the predictions of this analysis for an unstable initial density stratification, i.e., k and $B < 0$, which corresponds to $(\partial s/\partial y) > 0$, we see that a tendency towards an inflection point in the profile develops. This would indicate a tendency towards the development of an instability in the turbulent sense in agreement with physical intuition. The effects of stratification on skin friction can be displayed by the ratio of C_f to that without stratification at the same station, C_{f0}

$$C_f/C_{f_0} = 1 + 1/C^2(gk\nu x/U_e^3) = 1 + 1/C^2\,(Ri_x/Re_x) \tag{13}$$

Stokes' Flow Regime

In this flow regime, we neglect the inertia terms in comparison with the viscous and stratification terms in Eq. (1a) to give

$$\nabla^2(\nabla^2\psi) = \beta(\partial\psi/\partial x) \tag{14}$$

Where we have introduced $\beta \equiv (gk/U_e\nu)$. For unstratified flows, the equivalent approximation, studied by Stokes and others, simply has the right-hand side equal to zero. We note here that it is well known[5] that the Stokes' flow approximation for the planar flow of an unbounded stream past a body contains some mathematical difficulties far from the body in matching the usual separation of variables solution into the outer flow. The flow past a circular cylinder, for example, has been studied in detail by several workers. The difficulty is generally relieved by employing an Oseen approximation to the convective derivative in the far field.[5] One can expect analogous difficulties in the present instance. However, since the separation of variables approach is not applicable here so that a numerical solution procedure will be employed and only a finite flow regime will be considered, it is felt that Eq. (14) will provide a suitable basis for studying the relative effects of stratification. An Oseen approximation to Eq.

(1a) is easily written down as

$$U_e(\partial^3\psi/\partial y^2\partial x) + (gk/U_e)(\partial\psi/\partial x) = \nu\nabla^2(\nabla^2\psi) \quad (15)$$

and this could serve as the basis for an extended study.

Consider the flow past a flat plate of length L as governed by Eq. (14) in a region such that the leading edge of the plate is located at $x = x_0$, the upstream boundary is at $x = o$, the downstream boundary is at $x = x_1$ $[x_1 \gg (x_0 + L)]$ and the lateral boundaries are at $y = \pm H$ with the plate on $y = 0$.

The boundary conditions were chosen so as to approximately represent a plate in a uniform velocity stream

$$y = \pm H,\ 0 \le x \le x_1 \qquad \psi = \pm HU_e \quad (16a)$$
$$\partial\psi/\partial y = U_e \quad (16b)$$
$$\partial\psi/\partial x = 0 \quad (16c)$$
$$y = 0,\ x_0 \le x \le (x_0 + L) \qquad \psi = 0 \quad (16d)$$
$$\partial\psi/\partial y = 0 \quad (16e)$$
$$\partial\psi/\partial x = 0 \quad (16f)$$
$$x = 0 \text{ and } x_1,\ 0 \le |y| \le H \qquad \psi = U_e y \quad (16g)$$
$$\partial\psi/\partial x = 0 \quad (16h)$$
$$\partial\psi/\partial y = U_e \quad (16i)$$

For sufficiently large H, x_0 and x_1 compared to L, this approaches the situation for a uniform free stream of velocity U_e. However, as we shall see, the fact that the order of the equation forces us to specify ψ as well as the velocity at the lateral boundaries produces a lingering effect in the solutions. This is a fundamental problem since we cannot consider overly large regions numerically without prohibitive cost. A saving grace comes from the fact that we wish mainly to compare results for stratified and unstratified flows for the same physical problem, and the effects of the boundary conditions should not greatly influence that comparison.

Again, solutions are sought in the form

$$\psi = \psi^{(0)} + \beta\psi^{(1)} + (\beta^2/2!)\psi^{(2)} + \dots + (\beta^n/n!)\psi^{(n)}$$

For small β, only a few terms will be needed. To determine $\psi^{(i)}$, substitute into Eq. (14) and obtain

$$\nabla^4\psi^{(0)} = 0 \qquad \nabla^4\psi^{(1)} = +\psi_x^{(0)} \qquad \nabla^4\psi^{(2)} = +2\psi_x^{(1)}$$
$$\nabla^4\psi^{(n)} = +n\psi_x^{(n-1)} \quad (17)$$

where $\psi^{(0)}$ satisfies Eq. (16) and $\psi^{(i)}$, $i > 0$ satisfies zero boundary conditions. The heart of the problem is to solve an equation of the form

$$\nabla^4\psi = f(x,y) \quad (18)$$

Let

$$\nabla^2\psi = \Omega \quad (19)$$

Then Eq. (18) becomes

$$\nabla^2\Omega = f(x,y) \quad (20)$$

Superimpose a square grid over the region with mesh size $h = H/(N+1)$ for some integer N. Let

$$\psi(y,x) \equiv \psi(ih,jh) \equiv \psi_{i,j} \quad (21)$$

where

$$i = 0,\ \pm 1,\ \dots,\ \pm(N+1)$$
$$j = 0,\ 1,\ 2,\ \dots,\ M+1$$

We note

$$x_1 = [(M+1)/(N+1)]H$$

$$x_0 = Jh = JH/(N+1) \qquad x_0 + L = JLh = JL\cdot H/(N+1)$$

Replacing the differential equations by difference equations, we are led to

$$\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{ij} = h^2\Omega_{ij} \quad (22)$$

$$\Omega_{i+1,j} + \Omega_{i-1,j} + \Omega_{i,j+1} + \Omega_{i,j-1} - 4\Omega_{ij} = h^2 f_{ij} \quad (23)$$

from Eqs. (19) and (20)

$$\psi_{N+1,j} = HU_e \qquad j = 1,\ \dots,\ M$$
$$\psi_{-(N+1),j} = -HU_e \quad (24)$$

from Eq. (16a)

$$\psi_{i,o} = \psi_{i,M+1} = ihU_e \quad i = -N,\ \dots,\ N \quad (25)$$

from Eq. (16g)

$$\psi_{i,j} = 0 \qquad j = J,\ \dots,\ JL \quad (26)$$

from Eq. (16d)

$$\Omega_{N+1,j} = 1/2h^2\,[8\psi_{N,j} - \psi_{N-1,j} - 7HU_e + 6hU_e]$$
$$\Omega_{-(N+1),j} = 1/2h^2\,[8\psi_{-N,j} - \psi_{-(N-1),j} + 7HU_e - 6hU_e]$$
$$j = 1,\ \dots,\ M \quad (27)$$

from Eqs. (19) and (16a,b,c)

$$\Omega_{i,0} = 1/2h^2[8\psi_{i,1} - \psi_{i,2} - 7ihU_e]$$
$$i = -N,\ \dots,\ N \quad (28)$$
$$\Omega_{i,M+1} = 1/2h^2\,[8\psi_{i,M} - \psi_{i,M-1} - 7ihU_e]$$

from Eqs. (19) and (16g,h,i). The above conditions are for $\psi^{(0)}$. For $\psi^{(i)}$, $i > 0$ replace U_e with zero.

Since the plate is a boundary, we have two conditions along it, one above and one below. Above, we have

$$\Omega_{0,j} = 1/2h^2[8\psi_{1,j} - \psi_{2,j}] \qquad j = J+1,\ \dots,\ JL-1 \quad (29)$$

and below, we have

$$\Omega_{0,j} = 1/2h^2[8\psi_{-1,j} - \psi_{-2,j}] \qquad j = J+1,\ \dots,\ JL-1 \quad (30)$$

On the end points, we used

$$\Omega_{0,j} = 1/h^2[\psi_{1,J} + \psi_{-1,J} + \psi_{0,J-1}]$$
$$\Omega_{0,JL} = 1/h^2[\psi_{i,JL} + \psi_{-1,JL} + \psi_{0,JL+1}] \quad (31)$$

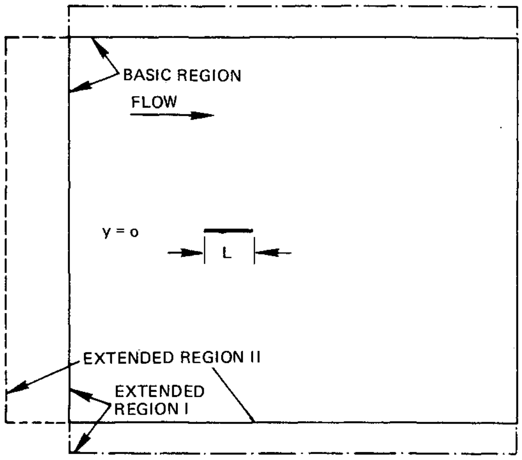

Fig. 3 Regions used for numerical calculations.

The iterative method used was a point Successive Over Relation (SOR) approach.[6] This is applied as follows. Let

$$\Omega_{i,j}^{(n)},\ \psi_{i,j}^{(n)} \tag{32}$$

be the nth iterants, where

$$\Omega_{i,j}^{(0)} = \psi_{i,j}^{(0)} = 0 \tag{33}$$

Then, we have the following sequence

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)} + \omega_1/4[\psi_{i+1,j}^{(n+1)} + \psi_{i-1,j}^{(n)} + \psi_{i,j-1}^{(n+1)} + \psi_{i,j+1}^{(n)} - 4\psi_{ij}^{(n)} - h^2\Omega_{i,j}^{(n)}] \tag{34}$$

$i = N, N-1, \ldots, \quad -N \; j = 1, \ldots, M$

$i = 0, \; j \neq J, J+1, \ldots, JL$

Equations (27-31) are used with $\psi_{i,j}^{(n+1)}$ to compute $\Omega_{i,j}^{(n+1)}$ on the boundaries

$$\Omega_{i,j}^{(n+1)} = \Omega_{i,j}^{(n)} + \omega_2/4[\Omega_{i+1,j}^{(n+1)} + \Omega_{i-1,j}^{(n)} + \Omega_{i,j-1}^{(n+1)} + \Omega_{i,j+1}^{(n)} - 4\Omega_{i,j}^{(n)} - h^2 f_{ij}] \tag{35}$$

$i = N, \; N-1, \ldots, \quad -N \; j = 1, \ldots, M$

$i = 0, \; j \neq J, \; J+1, \ldots, JL$

The constants ω_1 and ω_2 are relaxation factors chosen to speed up convergence. The same factors apply to each of the equations in Eq. (17) since the right sides are independent of the solution. An exact analysis of the iterative scheme is not available and some investigation for optimal ω's is called for. Convergence is reached when

$$\max\{|\psi_{ij}^{(n+1)} - \psi_{ij}^{(n)}|, \; |\Omega_{ij}^{(n+1)} - \Omega_{ij}^{(n)}|\} < \delta \tag{36}$$

for some δ. It should be pointed out that δ must be quite small, since ψ_x is to be computed and cancellation may produce numbers in round off range.

The solution for $\psi^{(0)}$ represents the flow of a homogeneous fluid past the plate and some calculations were made to study the influence of region size on the solution for this variable. In Fig. 3, are shown the location of the boundaries and the plate for the three cases considered. The stratified results to be presented below were obtained for the "Basic Region." The effect of moving the lateral boundaries outward (Extended Region I) and moving the upstream boundary further away from the plate (Extended Region II) are shown in Fig. 4 in terms of the velocity profile at a distance of $L/4$ from the leading edge of the plate. The first feature of these results that is noticeable is the existence of a velocity overshoot (i.e., $u > U_e$). This is directly caused by the specification of $\psi = \pm HU_e$ along the lateral boundary, $y = \pm H$, Thus, we are requiring the mass flow between the plate and the lateral boundary to be the same as that for a uniform flow in that region. Since viscous effects retard the flow near the plate, a velocity overshoot is required near the outer boundary. Iterative procedures to enable the specification of physically more realistic conditions on $\psi(x,H)$ can be envisioned, but we have not gone to that extra complication since our major goal has been to illustrate the effects of stratification. Secondly, it can be seen that it would be necessary to have a much wider region for the velocity overshoot to become negligible with $\psi(x,H) = HU_e$. Lastly, the further extension of the region in the horizontal direction has only a slight effect on the solution near the plate.

The solution for the first-order perturbation $u^{(1)}$ was obtained, and it is interesting to note that the perturbation is considerably larger before and behind the plate than in the region immediately adjacent to the plate. Further, the sign of the perturbation changes along the plate. For profiles before a station near the plate midpoint, the perturbation is negative near the axis and becomes positive with increasing y. Beyond that station, the behavior reverses.

Since the largest perturbations occur ahead of and behind the plate; detailed profiles showing the effect of the stratification are shown for one station in each of these regions in Figs. 5a and 5b. In Fig. 5a, we can see that the stratification works to make the wake behind the plate tend towards a uniform stream more rapidly. On the other hand, Fig. 5b shows that the upstream influence of the wake is increased.

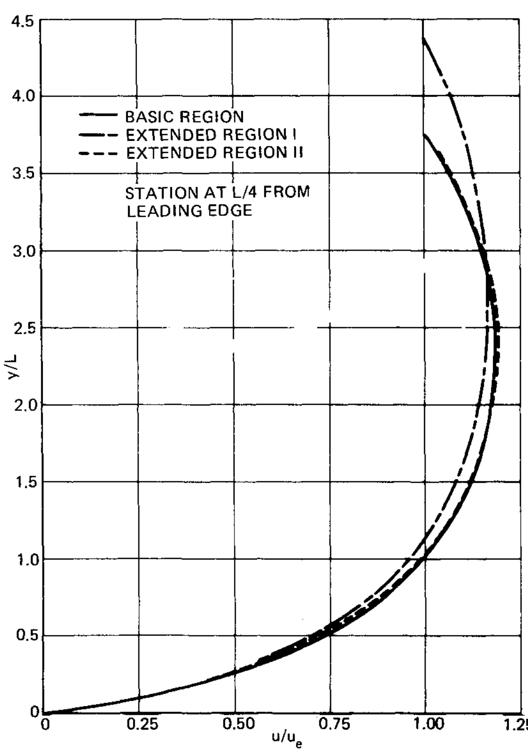

Fig. 4 Effect of region size and shape on numerical solution for unstratified fluid.

Far Wake behind the Plate

The far wake behind the plate has been analyzed using the boundary-layer form of the stratified flow equation [Eq. (1c)]. In this regime, the linearization

$$\psi_y(\partial/\partial x) - \psi_x(\partial/\partial y) \cong U_e(\partial/\partial x)$$

becomes rigorous since $u \approx U_e$ everywhere. This is to be distinguished from the heuristic approximation used in Eq. (6). The equation to be solved is, however, Eq. (1d) with $C \equiv 1$. A perturbation solution in terms of the parameter $Ri \equiv gkL^2/U_e^2$, where L is the length of the plate, is again sought.

The zeroth-order solution is that for the far wake behind a plate in a homogeneous fluid and that was developed by Tollmein[7] in 1931. In the present notation, that solution may be written

$$u^{(0)}/U_e = 1.000 - [0.664/(\pi)^{1/2}][Re(X+\alpha)]^{-1/2} \exp[-Y^2/4(X+\alpha)] \quad (37)$$

where

$$X \equiv (x/L)(U_e L/\nu)^{-1}, \quad Y \equiv y/L$$

and

$$\alpha \equiv (a/L)(U_e L/\nu)^{-1}$$

Also, x is measured in the downstream direction starting at a distance a behind the trailing edge of a plate of length L. This can be easily manipulated to give

$$\frac{\psi^{(0)}}{U_e L} = Y - 0.664\, Re^{-1/2} \operatorname{erf}\{Y/[4(X+\alpha)^{1/2}]\} \quad (38)$$

Before proceeding with a solution for $\psi^{(1)}$, we must consider boundary and initial conditions. We treat the following physical problem: In the region immediately behind the plate, stratification effects are negligible and the wake develops in accordance with Eq. (37); at a downstream distance equal to a, stratified effects begin to modify the wake behavior. This somewhat contrived physical problem may be justified on the basis of the following experimentally based observations. For high Reynolds' number flows, convective and viscous effects tend to dominate any effects due to stratification when the velocity variations are large. However, far behind a body where the velocity defect has become small, the effects of stratification begin to become important. Mathematically, this situation can be approximated by

$$\psi(0,Y) = \psi^{(0)}(0,Y)$$

$$\lim_{Y \to \pm\infty} \quad \psi(X,Y) = \pm Y \quad (39)$$

$$\partial^n\psi/\partial Y^n = 0 \text{ for } n > 1$$

which gives for the perturbation

$$\psi^{(1)}(0,Y) = 0$$

$$\lim_{Y \to \pm\infty} \psi^{(1)}(X,Y) = \partial^n x^{(1)}/\partial Y^n (X,Y) = 0 \quad (40)$$

The solution to this problem can be obtained by the method of infinite Fourier transforms. The work required is tedious but relatively straightforward and the solution for the velocity distribution can be written as

$$\frac{u^{(1)}(X,Y)}{U_e} = (4\pi^{3/2})\, 0.664\, Re^{-1/2} \frac{X}{(X+\alpha)^{1/2}} \exp[-Y^2/4(X+\alpha)] \quad (41)$$

Finally, then, the perturbation solution, to first order, of the wake behind a flat plate in a stratified fluid can be given by combining Eq. (41) and (37) as

$$\frac{u(x,y)}{U_e} = 1.000 - \frac{0.664}{(\pi)^{1/2}}\left(\frac{x+a}{L}\right)^{-1/2}\left[1 - 4\pi\left(\frac{x}{L}\right)\frac{Ri}{Re}\right] \exp\left[-\left(\frac{y}{L}\right)^2 Re/4\left(\frac{x+a}{L}\right)\right] \quad (42)$$

in more usual notation. Note, however, that $x = o$ at a

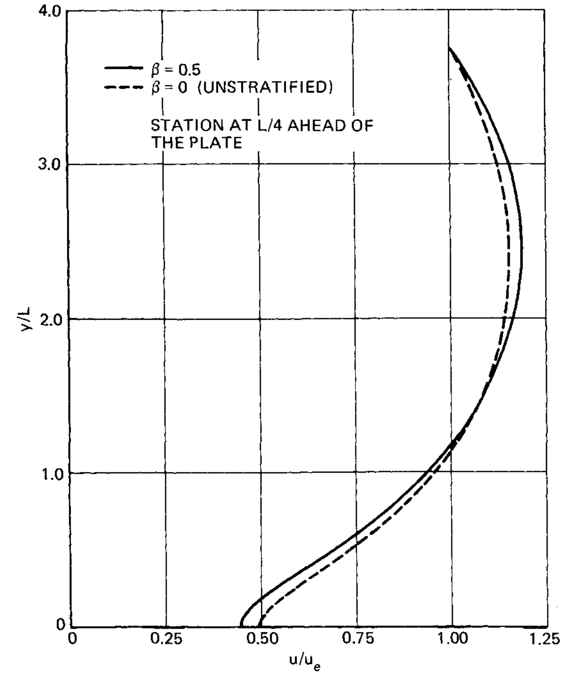

Fig. 5b Effect of stratification on the disturbance behind and before the plate.

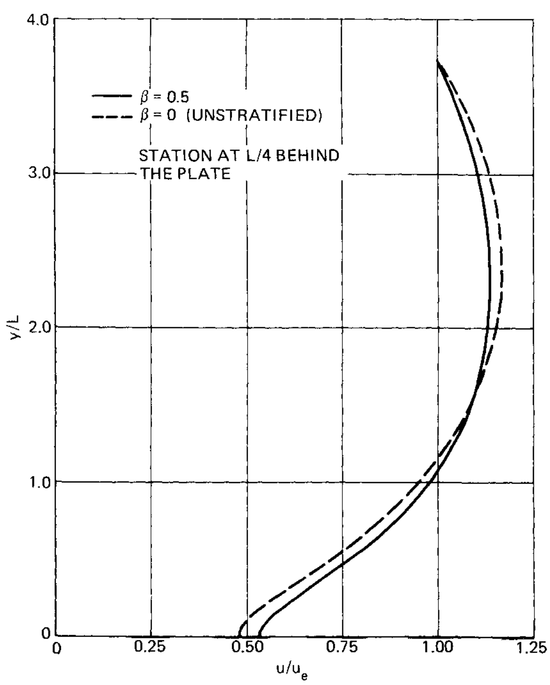

Fig. 5a Effect of stratification on the disturbance behind and before the plate.

distance a behind the plate. It must be remembered that this perturbation solution is only strictly valid in the range where $[4\pi(x/L)Ri/Re] \ll 1$.

It is interesting to examine this solution briefly. First, we can see that the major effect of the stratification ($Ri > 0$) is to increase the rate of decay of the velocity defect, $[U_e - u(o,y)]$. Second, this increased decay is linearly proportional to x and to the ratio of the Richardson to the Reynolds' numbers, Ri/Re. This analytical result can be useful in the design of scaled experiments since it indicates that the physical parameters are to be grouped as Re and Ri/Re.

Discussion

The mathematical formulation of planar, laminar, stratified flows has been considered in some detail. Suitable approximations of the exact equation for the nondiffusive case were investigated, and analytical results based on two of these approximations for the simple problem of a finite, flat plate have been presented.

The results for the boundary-layer type of approximation, in particular, indicate some interesting effects that can be of practical significance. The predicted increase in wall shear on the plate and the form of the increased decay of the velocity defect in the wake should be the subject of careful experimental study.

References

[1] Yih, C. S., "Stratified Flows," *Annual Reviews of Fluid Mechanics*, Vol. 1, 1969, pp. 73–110.

[2] Martin, S. and Long, R. R., "The Slow Motion of a Flat Plate in a Viscous Stratified Flow," *Journal of Fluid Mechanics*, Vol. 31, Pt. 4, 1968, pp. 669–688.

[3] Pao, Y-H., "Laminar Flow of a Stably Stratified Fluid Past a Flat Plate," *Journal of Fluid Mechanics*, Vol. 34, Pt. 4, 1968, pp. 795–808.

[4] Schetz, J. A. and Jannone, J., "Linearized Approximations to the Boundary Layer Equations," *Transactions of the ASME, Ser. E: Journal of Applied Mechanics*, Vol. 87, No. 4, Dec. 1965, pp. 757–764.

[5] Lagerstrom, P. A., "Laminar Flow Theory," *Theory of Laminar Flows; High Speed Aerodynamics and Jet Propulsion*, Vol. IV, edited by F. K. Moore, Princeton University Press, Princeton, N.J., 1964, Chap. B, p. 84.

[6] Ehrlich, L. W., "Point and Block SOR Applied to a Coupled Set of Difference Equations," TG 1166, July 1971, Applied Physics Lab., Johns Hopkins Univ., Silver Spring, Md.

[7] Tollmein, W., "Grenzschichten," Handbuch der Experimental-Physik, Vol. IV, Pt. 1, 1931, pp. 241–287.